

Cohomology of groups with normal Engel subgroups

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Abstract

The paper contains several results on the structure of 1-cocycles and the first cohomology group for a pair of representations of a connected group with a normal Engel subgroup.

1 Introduction

Through the paper G is a *connected locally compact group*.

Let π be a representation of G on a Hilbert space X and L be a π -invariant subspace of X . Set $\mathfrak{H} = L^\perp$. Denote by $B(\mathfrak{H}, L)$ the space of all bounded operators from \mathfrak{H} to L , $B(\mathfrak{H}) = B(\mathfrak{H}, \mathfrak{H})$ and $B(L) = B(L, L)$. Then

$$X = L \oplus \mathfrak{H} \text{ and } \pi(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & U(g) \end{pmatrix} \text{ for } g \in G, \quad (1.1) \quad \boxed{1}$$

where λ and U are representations of G on L and \mathfrak{H} , respectively, and the map $\xi: G \rightarrow B(\mathfrak{H}, L)$ is a (λ, U) *cocycle*, i.e.,

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)U(h) \text{ for } g, h \in G.$$

It is a (λ, U) -*coboundary* if $\xi(g) = \lambda(g)T - TU(g)$ for all $g \in G$ and some $T \in B(\mathfrak{H}, L)$.

The representation π is called an *extension of λ by U* . If ξ in (1.1) is (λ, U) -coboundary then there is a π -invariant subspace H such that $X = L \dot{+} H$, so that L has a π -invariant complement.

Denote by $\mathcal{Z}(G, \lambda, U)$ the set of all (λ, U) -cocycles and by $\mathcal{B}(G, \lambda, U)$ the set of all (λ, U) -coboundaries. Then $\mathcal{H}^1(G, \lambda, U) = \mathcal{Z}(G, \lambda, U)/\mathcal{B}(G, \lambda, U)$ is the *first cohomology group* of G related to the representations λ, U . If each (λ, U) -cocycle is a (λ, U) -coboundary then $\mathcal{H}^1(G, \lambda, U) = 0$. If $\dim L = 1$ and the representation $\lambda = \iota$ is the trivial representation on L ($\iota(g) \equiv \mathbf{1}_L$), then $\mathcal{H}^1(G, \iota, U)$ is the standard cohomology group of U (see [G]).

Maps $\lambda: G \rightarrow B(L)$ and $U: G \rightarrow B(\mathfrak{H})$ are *spectrally disjoint* at $h \in G$ (see [KS1]), if

$$\mathrm{Sp}(\lambda(h)) \cap \mathrm{Sp}(U(h)) = \emptyset.$$

In [KS1] we investigated the cohomology group $\mathcal{H}^1(G, \lambda, U)$ in the case when G is an Engel group (in particular, a nilpotent group). We showed in Corollaries 2.9 and 2.16 there that if λ and U are spectrally, or sectionally spectrally disjoint then $\mathcal{H}^1(G, \lambda, U) = 0$. We applied this to describe extensions of the elementary representations λ of N by unitary representations U (see (1.1)).

In this paper we consider the much more general case that G contains a *connected normal subgroup* N and impose conditions on the restrictions π^N, ξ^N, λ^N and U^N of π, ξ, λ and U to N .

We show in Corollary 2.3 that if 1) U has an upper triangular form and λ is spectrally disjoint with each diagonal U_i at some $h^i \in N$, and 2) $\mathcal{H}^1(N, \lambda^N, U^N) = 0$, then $\mathcal{H}^1(G, \lambda, U) = 0$. If N is an Engel group and U has a *finite* upper triangular form, then $\mathcal{H}^1(G, \lambda, U) = 0$ if only condition 1) holds (see Corollary 2.4).

In Theorem 2.5 we describe the structure of (λ, U) -cocycles in the case when 1) $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i$, where all \mathfrak{H}_i are U -invariant, 2) λ is spectrally disjoint with each $U_i = U|_{\mathfrak{H}_i}$ at some $h^i \in N$ and 3) $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ for $i \in \mathbb{N}$.

Let $\lambda = \iota$ be the trivial representation of G . Then the description of all (ι, U) -cocycles is given in Corollary 2.7, where condition 2) above is replaced by the following condition: 2') $1 \notin \text{Sp}(U_i(h^i))$ for all i and some $h^i \in N$. If, in addition, N is an Engel group, then condition 3) above holds automatically. Finally, if $G = N$ is a nilpotent group, then all conditions 1), 2'), 3) in Corollary 2.7 can be replaced by only one condition: $\mathfrak{H}_e := \{x \in \mathfrak{H} : U(g)x = x \text{ for all } g \in G\} = \{0\}$ (see Theorem 2.10).

2 Spectrally disjoint representations. Trivial cohomology groups.

For $g \in G$ and $h \in N$, we have $gh = h_g g$, where $h_g = ghg^{-1} \in N$. The following lemma is an important tool in the proofs of further results.

L1.1 **Lemma 2.1** *Let a map $\mu: G \rightarrow B(L)$ and a representation $U: G \rightarrow B(\mathfrak{H})$ be spectrally disjoint at some $h \in N$. Then $\text{Sp}(\lambda(h)) \cap \text{Sp}(U(h_g)) = \emptyset$ for each $g \in G$. So*

- 1) *If $\mu(h)Y = YU(h_g)$ for some $Y \in B(\mathfrak{H}, L)$ and some $g \in G$, then $Y = 0$.*
- 2) *If $Y\mu(h) = U(h_g)Y$ for some $Y \in B(L, \mathfrak{H})$ and some $g \in G$, then $Y = 0$.*

Let U be a representation of G on \mathfrak{H} and let $\{H_i\}_{i=0}^k$, $H_0 = \{0\}$ and $k \leq \infty$, be a nest of U -invariant subspaces of \mathfrak{H} : $H_i \subset H_{i+1}$ and $\mathfrak{H} = \bigcup_i H_i$. Set $\mathfrak{H}_i = H_{i+1} \ominus H_i$ and let U_i be the representation on \mathfrak{H}_i generated by U . Then

$$\mathfrak{H} = \sum_{i=1}^k \oplus \mathfrak{H}_i \text{ and } U = \begin{pmatrix} U_1 & U_{12} & U_{13} & \cdots \\ 0 & U_2 & U_{23} & \cdots \\ 0 & 0 & U_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \text{ We write } U = [U_i]_{i=1}^k. \quad (2.1) \quad \text{h2.5}$$

Denote by λ a representation of G on a Hilbert space L , and by λ^N and U^N the restrictions of λ and U to the subgroup N .

P2.1 **Proposition 2.2** *Let λ be spectrally disjoint with each U_i at some $h^i \in N$. Suppose that the restriction ξ^N of a (λ, U) -cocycle ξ to N is a (λ^N, U^N) -coboundary, i.e.,*

$$\xi(h) = \lambda(h)T - TU(h) \text{ for all } h \in N \text{ and some } T \in B(\mathfrak{H}, L).$$

Then ξ is a (λ, U) -coboundary: $\xi(g) = \lambda(g)T - TU(g)$ for $g \in G$, and T is a unique such operator.

Proposition 2.2 yields

C2.1 **Corollary 2.3** Let $U = [U_i]_{i=1}^k$ and λ be spectrally disjoint with each U_i at some $h^i \in N$.

- (i) If $\mathcal{H}^1(N, \lambda^N, U^N) = 0$ then $\mathcal{H}^1(G, \lambda, U) = 0$.
- (ii) If $k < \infty$ and all $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$, $i = 1, \dots, k$, then $\mathcal{H}^1(G, \lambda, U) = 0$.

In Corollary 2.3(ii) two conditions are required:

- 1) spectral disjointness of λ and all U_i ;
- 2) $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$, $1 \leq i \leq k < \infty$.

We will consider now the case when only condition 1) is needed.

Each $h \in G$ defines a map ad_h on G : $\text{ad}_h(g) = ghg^{-1}h^{-1}$ for $g \in G$. We say that h is an *Engel element* if, for each $g \in G$, there is $n_g \in \mathbb{N}$ such that $\text{ad}_h^{n_g}(g) = e$; G is an *Engel group*, if it consists of Engel elements. Nilpotent groups are Engel groups, while solvable ones are not always Engel.

C2 **Corollary 2.4** Let $U = [U_i]_{i=1}^k$, $k < \infty$. If λ is spectrally disjoint with each U_i at Engel element h_i of N , then $\mathcal{H}^1(G, \lambda, U) = 0$.

For $k = \infty$, the condition that all $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ is not sufficient for $\mathcal{H}^1(G, \lambda, U) = 0$.

We consider now a particular case of decomposition (2.1). Using Proposition 2.2, we have

T1 **Proposition 2.5** Let λ, U be representations of G on L and \mathfrak{H} , respectively. Let $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i$, where all \mathfrak{H}_i are U -invariant. Set $U_i = U|_{\mathfrak{H}_i}$. Suppose that 1) λ be spectrally disjoint with each U_i at some $h^i \in N$; and 2) $\mathcal{H}^1(N, \lambda^N, U_i^N) = 0$ for $i \in \mathbb{N}$. Then

- (i) $\mathcal{H}^1(G, \lambda, U_i) = 0$ for all $i \in \mathbb{N}$.

- (ii) A map $\xi: G \rightarrow B(\mathfrak{H}, L)$ is a (λ, U) -cocycle if and only if there are unique operators $T_i \in B(\mathfrak{H}_i, L)$ such that

$$\xi(g)Q_i = \lambda(g)T_i - T_iU_i(g) \text{ for all } g \in G \text{ and } i \in \mathbb{N},$$

where Q_i are the projections on \mathfrak{H}_i . It is a (λ, U) -coboundary if and only if $T = \{T_i\}_{i=1}^{\infty} \in B(\mathfrak{H}, L)$.

Let $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i$ and all \mathfrak{H}_i be U^N -invariant. Below we give some conditions for all \mathfrak{H}_i to be also U -invariant.

P4.1 **Proposition 2.6** Let U be a weakly continuous representation of G on \mathfrak{H} and let $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i$, where all \mathfrak{H}_i are invariant for U^N . If $U^N|_{\mathfrak{H}_i} = \omega_i \mathbf{1}_{\mathfrak{H}_i}$ for some distinct unitary characters $\{\omega_i\}_{i=1}^{\infty}$ on N , then all \mathfrak{H}_i are U -invariant.

Let now $\dim L = 1$, i.e., $L = \mathbb{C}e$. Then each operator in $B(\mathfrak{H}, L)$ has form $z \otimes e$, where $(z \otimes e)x = (x, z)_{\mathfrak{H}}e$ for all $x \in \mathfrak{H}$ and some $z \in \mathfrak{H}$. Every map $\xi: G \rightarrow B(\mathfrak{H}, L)$ has form $\xi(g) = r(g) \otimes e$ for some map $r: G \rightarrow \mathfrak{H}$. Each $T \in B(\mathfrak{H}_i, L)$ has form $T = y \otimes e$ for some $y \in \mathfrak{H}_i$.

If $\lambda = \iota$ is the trivial representation of G on L , then Proposition 2.5 has a simpler form.

C3.1n **Corollary 2.7** Let U be a representation of G on \mathfrak{H} and $\mathfrak{H} = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i$, where all \mathfrak{H}_i are U -invariant. Set $U_i = U|_{\mathfrak{H}_i}$. Let 1) $1 \notin \text{Sp}(U_i(h^i))$ for all i and some $h^i \in N$, and let 2) $\mathcal{H}^1(G, \iota^N, U_i^N) = 0$ for all $i \in \mathbb{N}$. Then

- (i) $\mathcal{H}^1(G, \iota, U_i) = 0$ for all $i \in \mathbb{N}$.

(ii) A map $\xi: G \rightarrow r(g) \otimes e$ is a (ι, U) -cocycle if and only if there are unique $y_i \in \mathfrak{H}_i$ such that

$$r(g) = \oplus_{i=1}^{\infty} (y_i - U_i^*(g)y_i) \in \mathfrak{H}, \text{ i.e., } \sum_{i=1}^{\infty} \|y_i - U_i^*(g)y_i\|^2 < \infty \text{ for } g \in G. \quad (2.2) \quad \boxed{1.5}$$

It is a (ι, U) -coboundary if and only if the operator $(y_i \otimes e)_{i=1}^{\infty} \in B(\mathfrak{H}, L)$, i.e., $\sum_{i=1}^{\infty} \|y_i\|^2 < \infty$.

If N has property (T) (ι is isolated in the set \widehat{N} of all irreducible representations of N), then $\mathcal{H}^1(N, \iota, V) = \{0\}$ for any unitary representation V [G]. So Corollary 2.7 yields

C2.5 **Corollary 2.8** Let G, U and \mathfrak{H} be as in Corollary 2.7 and U^N be unitary. Let 1) $1 \notin \text{Sp}(U_i(h^i))$ for all i and some $h^i \in N$, and 2) N have property (T). Then the results of Corollary 2.7 hold.

Using Corollary 2.4, we obtain the following generalization of Corollary 2.7 without the assumption that N has property (T).

C2.6 **Corollary 2.9** Let G, U and \mathfrak{H} be as in Corollary 2.7. If $1 \notin \text{Sp}(U_i(h^i))$ for all i and some Engel elements $h^i \in N$, then the results of Corollary 2.7 hold.

Let $G = N$ be a nilpotent group and U be a unitary representation of N on \mathfrak{H} . Assume that $\mathfrak{H}_e := \{x \in \mathfrak{H}: U(g)x = x \text{ for all } g \in G\} = \{0\}$. Then it follows from Proposition 3.3 [KS1] that there is a decomposition $\mathfrak{H} = \oplus_{i=1}^k \mathfrak{H}_i$, $k \leq \infty$, such that all \mathfrak{H}_i are U -invariant and the representation ι is spectrally disjoint with each $U_i = U|_{\mathfrak{H}_i}$ at some $h^i \in N$, i.e., $1 \notin \text{Sp}(U_i(h^i))$. Therefore, since N is an Engel group, Corollary 2.9 yields

T2.5 **Theorem 2.10** Let N be a nilpotent connected group and U be a unitary representation of N on \mathfrak{H} . If $\mathfrak{H}_e = \{0\}$ then there is a decomposition $\mathfrak{H} = \oplus_{i=1}^k \mathfrak{H}_i$, $k \leq \infty$, such that all \mathfrak{H}_i are U -invariant, $\mathcal{H}^1(N, \iota, U_i) = 0$ for all i and (2.2) describes (ι, U) -cocycles.

Let U be a unitary representation of G on \mathfrak{H} . Let a normal subgroup N of G be nilpotent, connected and locally compact. By Theorem 2.10, $\mathfrak{H} = \oplus_{i=1}^k \mathfrak{H}_i$ for $k \leq \infty$, where \mathfrak{H}_i are U^N -invariant subspaces, and $\mathcal{H}^1(N, \iota, U_i^N) = 0$ for all i . If $U^N|_{\mathfrak{H}_i} = \omega_i \mathbf{1}_{\mathfrak{H}_i}$ for some distinct unitary characters $\{\omega_i\}_{i=1}^k$, then by Proposition 2.6, all \mathfrak{H}_i are U -invariant and $\mathcal{H}^1(G, \iota, U_i) = 0$ for all i .

Some applications of the obtained results to spectral continuity and representations in the spaces with indefinite metric will be considered in subsequent publications.

References

- G** [G] Guichardet A., Cohomologie des groupes topologiques et des algebres de Lie, Cedic/Fernand Nathan, Paris, 1980.
- KS1** [KS1] Kissin E. and Shulman V.S., Non-unitary representations of nilpotent groups, I: cohomologies, extensions and neutral cocycles, J. Func. Anal., **269**(2015), 2564-2610. DOI: 10.1016/j.jfa.2015.07.003

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